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# MAGNETOGENESIS, VARIATION OF GAUGE COUPLINGS AND INFLATION

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The interplay among the possible variation of gauge coupling and the inflationary dynamics is investigated in a simplified toy model. Depending upon various parameters (scalar mass, curvature scale at the end of inflation and at the onset of the radiation epoch), the two-point function of the magnetic inhomogeneities grows during the de Sitter stage of expansion and, consequently, large scale magnetic fields are generated. The requirements coming from inflationary magnetogenesis are examined together with the theoretical constraints stemming from the explicit model of the evolution of the gauge coupling.

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# 1 Introduction

Suppose that during a de Sitter stage of expansion the coupling constant of an Abelian gauge field evolves in time. Thus the kinetic term of the gauge field can be written (in four space-time dimensions) as

$$S_{\rm em} = -\frac{1}{4}\int d^4 x\sqrt{-G} f(\phi) F_{\alpha\beta}F^{\alpha\beta}, \tag{1.1}$$

where $G$ is the determinant of the space-time metric, $\phi$ is a scalar field (which can depend upon space and time) and $g(\phi) = f(\phi)^{-1/2}$ is the coupling [2]. This type of vertex is typical of scalar-tensor theories of gravity [1] and of the low energy string effective action [2]. Early suggestions that the Abelian gauge coupling may change over cosmological times were originally made by Dirac [3] (and subsequently discussed in [4] and in [5]) mainly in the framework of ordinary electromagnetism.

The dynamics of the field $\phi$ will be described by the action of a minimally coupled (massive) scalar. If the field is displaced from the minimum of its potential during inflation, there will be a phase where the field relaxes. Provided the scalar mass is much smaller than the curvature scale during inflation such a phase could be rather long. During the de Sitter stage, the specific form of the expanding background will dictate, through the equations of motion, the rate of suppression of the amplitude of $\phi$.

While the field relaxes toward the minimum of its potential, energy is pumped from the homogeneous mode of $\phi$ to the gauge field fluctuations. The function $f(\phi)$ can be either an increasing function of $\phi$ (leading to a decreasing coupling) or a decreasing function of $\phi$ (leading to an increasing coupling). In both cases, depending upon the parameters of the model, the two-point correlation function of magnetic inhomogeneities increases during the inflationary stage. This implies that large scale magnetic fields can be potentially generated.

The implications of large scale magnetic fields for cosmology, astrophysics and high-energy physics have been recently reviewed in [6] where various useful references are reported. The interested student will find in [6] a morte physical introduction to the subject of large scale magnetic fields. The toy model discussed in the present paper is only an useful exercise suggesting a possible connection between the variation of gauge couplings and the production of large scale magnetic fields.

Recently [9] the variation in the fine structure constant has been suggested by observations of absorption lines at different frequencies. The results of these claims can be summarized by saying that $\alpha_{\rm em}$ was smaller in the past. Defining, indeed $\Delta\alpha_{\rm em} = \alpha_{\rm today} - \alpha_{\rm past}$, measurements suggest that

$$\frac{\Delta\alpha_{\rm em}}{\alpha_{\rm em}} = (0.72 \pm 0.18)10^{-5}. \tag{1.2}$$

[2] The Heaviside electromagnetic system of units will be used throughout the investigation. The effective "electron" charge will then be given, in the present context, by $e(\phi) = e_1 f(\phi)^{-1/2}$.

To avoid confusions, in the present scenario, gauge couplings are *constant* today and their variation occurs prior to big-bang nucleosynthesis (BBN).

The gauge coupling has to be (almost) constant by the time when the Universe is approximately old of one second, namely by the time of BBN. The abundances of light elements are very sensitive to any departure from the standard cosmological model. Hence, fluctuations in the baryon to photon ratio, matter–antimatter domains, anisotropies in the expansion of the four space-time dimensions, can all be successfully constrained by demanding that the abundances of the light elements are correctly reproduced. Following the same logic the variation in the gauge couplings can also be constrained from BBN [10].

The plan of the present contribution is then the following. In Section II the basic ideas concerning the model of evolution of the gauge coupling will be introduced. In Section III bounds coming both from the homogeneous and from the inhomogeneous evolution of $\phi$ will be described. In Section IV the evolution of the magnetic inhomogeneities will be addressed along the various stages of the model with particular attention to the role of the two-point function. In Section V the large scale magnetic fields produced in the scenario will be estimated. Section VI contains some concluding remarks. The considerations reported in the present paper extend and complement the results of [11].

Finally, in concluding this Introduction, I wish to remind a couple of useful references of two other lecturers of the Chalonge school. In Ref. [7] an interesting discussion on the possible ambiguities arising in the observations of large scale fields in clusters is reported. In [8] another idea on the generation of magnetic fields is reported.

# 2 Basic Equations

Thanks to the high degree of isotropy and homogeneity of the observed Universe, the background geometry can be described using a (conformally flat) Friedmann-Robertson-Walker (FRW) line element

$$ds^2 = G_{\mu\nu} dx^{\mu} dx^{\nu} = a^2(\eta)[d\eta^2 - d\vec{x}^2], \tag{2.1}$$

where $\eta$ is the conformal time coordinate and $G_{\mu\nu}$ is the four-dimensional space-time metric. The cosmic time coordinate (often employed in this investigation) is related to $\eta$ as $a(\eta)\ d\eta = dt$.

From the anisotropies of the Cosmic Microwave Background (CMB) it is consistent to assume that the Universe underwent a period of inflationary expansion of de Sitter or quasi-de Sitter type. Therefore $a(\eta) \sim -\eta_1/\eta$ during a phase stopping, approximately, when the curvature scale was $H_1 \leq 10^{-6} M_{\rm P}$. For $\eta > \eta_1$ (possibly after a transient period) the Universe gets dominated by radiation [i.e. $a(\eta) \sim \eta$] and then, after decoupling, by dust matter [i.e. $a(\eta) \sim \eta^2$].

The action describing the dynamics of the (Abelian) gauge coupling in a given background geometry can be parametrized as

$$S = \int d^4x\,\sqrt{-G}\biggl[\frac{1}{2}G^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - V(\phi) - \frac{1}{4g^2(\phi)}F_{\mu\nu}F^{\mu\nu}\biggr]. \tag{2.2}$$

In Eq. (2.2) $\phi$ *is not the inflaton* but an extra (scalar) degree of freedom taking care of the evolution of the gauge coupling.

From Eq. (2.2) the equations of motion can be derived

$$\frac{1}{\sqrt{-G}}\partial_{\mu}\biggl[\sqrt{-G}G^{\mu\nu}\partial_{\nu}\phi\biggr] + \frac{\partial V}{\partial\phi} = \frac{1}{2g^3(\phi)}\frac{\partial g}{\partial\phi}F_{\alpha\beta}F^{\alpha\beta}, \tag{2.3}$$

$$\frac{1}{\sqrt{-G}}\partial_{\alpha}\biggl[\frac{\sqrt{-G}}{g^2(\phi)}F^{\alpha\beta}\biggr] = 0. \tag{2.4}$$

If the gauge coupling does not change, the evolution of Abelian gauge fields is conformally invariant. Hence, using the the conformal time coordinate, the appropriately rescaled gauge field amplitudes obey a set of equations which is exactly the one they would obey Minkowski space. In order to simplify the explicit form of the equations of motion the electric and magnetic fields will be rescaled in such a way that the obtained system of equations will reproduce the usual (conformally invariant) system in the limit $g \to$ constant. The rescalings in the fields are

$$\vec{B} = a^2\vec{\mathcal{B}},\quad \vec{E} = a^2\vec{\mathcal{E}},\qquad \vec{A} = a\vec{\mathcal{A}}, \tag{2.5}$$

where $\vec{\mathcal{B}}$, $\vec{\mathcal{E}}$, $\vec{\mathcal{A}}$, are the flat-space quantities whereas $\vec{B}$, $\vec{E}$, $\vec{A}$, are the curved-space ones.

The explicit form of our system becomes, in the metric (2.1):

$$\frac{\partial^2\phi}{\partial\eta^2} + 2\mathcal{H}\frac{\partial\phi}{\partial\eta} + \frac{\partial V}{\partial\phi}a^2 - \nabla^2\phi = \frac{1}{2g^3a^2}\frac{\partial g}{\partial\phi}\biggl[\vec{E}^2 - \vec{B}^2\biggr], \tag{2.6}$$

$$\frac{\partial\vec{B}}{\partial\eta} = -\vec{\nabla}\times\vec{E}, \tag{2.7}$$

$$\frac{\partial}{\partial\eta}\biggl[\frac{1}{g^2(\phi)}\vec{E}\biggr] = \frac{1}{g^2(\phi)}\biggl[\vec{\nabla}\times\vec{B} - \frac{2}{g}\frac{\partial g}{\partial\phi}\vec{\nabla}\phi\times\vec{B}\biggr], \tag{2.8}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{2.9}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{2}{g}\frac{\partial g}{\partial\phi}\vec{E}\cdot\vec{\nabla}\phi, \tag{2.10}$$

where $\vec{v}$ is the bulk velocity of the plasma. The spatial gradients used in Eqs. (2.6)–(2.10) are defined according to the metric (2.1). In Eq. (2.11) the quantity $\mathcal{H} = \partial\ln a/\partial\eta$ has also been introduced. $\mathcal{H}$ is the Hubble factor in conformal time which is related to the Hubble factor in cosmic time as $\mathcal{H} = H/a$ where $H = \partial\ln a/\partial t$.

Once the background geometry is specified we are interested in the situation when the gauge field background is vanishing and the only fluctuations are the ones associate with the vacuum state of the Abelian gauge fields. Hence, Eqs. (2.6)–(2.10) allow to compute the evolution of $\phi$ and the associated evolution of the two-point function of the gauge field fluctuations.

Suppose that $\phi$ is originally displaced from the minimum of its potential. As far as the zero mode of $\phi$ is concerned the system of equations can be further simplified:

$$\frac{\partial^2\phi}{\partial\eta^2} + 2\mathcal{H}\frac{\partial\phi}{\partial\eta} + \frac{\partial V}{\partial\phi}a^2 = \frac{1}{2g^3a^2}\frac{\partial g}{\partial\phi}\Big[\vec{E}^2 - \vec{B}^2\Big], \tag{2.11}$$

$$\frac{\partial\vec{B}}{\partial\eta} = -\vec{\nabla}\times\vec{E}, \tag{2.12}$$

$$\frac{\partial}{\partial\eta}\Big[\frac{1}{g^2(\phi)}\vec{E}\Big] = \frac{1}{g^2(\phi)}\vec{\nabla}\times\vec{B}, \tag{2.13}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\cdot\vec{E} = 0, \tag{2.14}$$

By now combining together the modified Maxwell's equations we obtain the evolution of the magnetic fields

$$\vec{B}'' - 2\frac{g'}{g}\vec{B}' - \nabla^2\vec{B} = 0, \tag{2.15}$$

where the prime denoted derivation with respect to the conformal time coordinate (the over-dot will denote, instead, derivation with respect to cosmic time).

Once the evolution of the metric is specified, Eq. (2.11) dictates a specific evolution for $\phi$ and the evolution of $\phi$ will determine, in its turn, the evolution of the gauge fields. The interesting initial conditions for the system are the ones where the classical gauge field background vanishes. Thus, when the homogeneous component of $\phi$ starts its evolution during the de Sitter phase, quantum mechanical fluctuations will be postulated as initial conditions of gauge inhomogeneities.

When the background geometry evolves from the de Sitter phase to the subsequent epoch, massive quanta of $\phi$ are produced. The amount of the produced inhomogeneous modes of $\phi$ can be computed and it will be shown that the associated energy density will always be smaller than the one of the homogeneous mode. This analysis will be one of the subjects discussed in Section III.

# 3 Constraints on the evolution of the gauge coupling

Sub-millimiter tests of the Newton's law show that no deviations are observed down to distances as small as 0.1 mm [12]. Therefore, $\phi$ should be massive. Of course the potential of $\phi$ may be much more complicated than the one provided by a simple mass term. However, for sake of simplicity, a massive scalar will be analyzed since, already in this case, interesting effects can be analyzed. In spite of the fact that this choice is apparently simple, various constraints on the scalar mass appear.

## 3.1 Evolution of the homogeneous mode

Suppose that the potential term driving the evolution of the gauge coupling is simply

$$V(\phi) \sim \frac{m^2}{2}\phi^2. \tag{3.1}$$

During the inflationary stage of expansion the scale factor evolves as $a(\eta) = (-\eta_1/\eta)$ for $\eta < -\eta_1$. The evolution of $\phi$ is obtained by solving

$$\phi'' + \frac{2}{\eta}\phi' + \frac{\mu^2}{\eta^2}\phi = 0, \tag{3.2}$$

where $\mu = m/H_1$ and where the relation $\mathcal{H} \sim \eta^{-1} \sim aH$ has been used. If $\mu \ll 1$, for $\eta < -\eta_1$ the solution of Eq. (3.2) can be written as

$$\phi_{\rm i}(\eta) = \phi_1 - \phi_2\Big(-\frac{\eta}{\eta_1}\Big)^3. \tag{3.3}$$

The end of the inflationary stage of expansion may not be directly followed by the radiation dominated phase. In the intermediate phase the scalar mass is still small than the curvature scale but the curvature decreases, in general, faster than during the inflationary phase since the background is neither of de Sitter nor of quasi-de Sitter type. The evolution of the scale factor can be parametrized as $a(\eta) \sim \eta^{\alpha}$ where, in order to fix the ideas, $\alpha \sim 2$ could be assumed [3]. The evolution of $\phi$ will simply be

$$\phi_{\rm rh}(\eta) = \phi_1 + \phi_2\biggl[\frac{2\alpha - 4}{2\alpha - 1} + \frac{3}{2\alpha - 1}\Big(\frac{\eta}{\eta_1}\Big)^{2\alpha - 1}\biggr], \quad \eta_1 < \eta < \eta_{\rm r}, \tag{3.4}$$

where the continuity between Eq. (3.3) and Eq. (3.4) has been required, so that $\phi_{\rm i}(-\eta_1) = \phi_{\rm rh}(\eta_1)$ and $\phi_{\rm i}'(-\eta_1) = \phi_{\rm rh}'(\eta_1)$.

After $\eta_{\rm r}$ the background enters a radiation dominated phase and the evolution of $\phi$ can be explicitly solved in cosmic time. The equation for $\phi$ , in this phase, is given by

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0, \quad H = \frac{\dot{a}}{a}, \tag{3.5}$$

which in terms $\Phi = a^{\frac{3}{2}}\phi$, becomes

$$\ddot{\Phi} + \biggl[m^2 - \frac{3}{2}\dot{H} - \frac{9}{4}H^2\biggr]\Phi = 0. \tag{3.6}$$

[3] The case $\alpha = 2$ corresponds to a matter-dominated intermediate stage.

In the radiation-dominated stage of expansion Eq. (3.6) becomes

$$\ddot{\Phi} + \left[m^2 + \frac{3}{16t^2}\right]\Phi = 0, \tag{3.7}$$

whose solution can be written in terms of Bessel functions [13]

$$\Phi(mt) = \sqrt{mt}\left[AY_{\frac{1}{4}}(mt) + BJ_{\frac{1}{4}}(mt)\right]. \tag{3.8}$$

For $mt \ll 1$, $\phi$ has a constant mode and a solution as $t^{-1/2}$. Recalling the relation between cosmic and conformal time and imposing the continuity of $\phi$ and $\phi'$ (in $\eta_r$) with the solution of Eq. (3.4) the following form can be obtained:

$$\phi_{\rm r}(\eta) = \phi_1 + \phi_2\left[\left(\frac{2\alpha-4}{2\alpha-1} + \frac{6(1-\alpha)}{2\alpha-1}\left(\frac{\eta_1}{\eta_{\rm r}}\right)^{2\alpha-1}\right) + 3\left(\frac{\eta_1}{\eta_{\rm r}}\right)^{2\alpha-1}\frac{\eta_{\rm r}}{\eta}\right], \tag{3.9}$$

which is valid for $\eta_r < \eta < \eta_{\rm m}$. The time $\eta_{\rm m}$ marks the moment where $H \sim m$. When $mt > 1$, the regime of coherent oscillations takes over and the solution (3.8) implies that the energy density stored in $\phi$ decreases as $a^{-3}$, meaning that $\phi_{\rm c}(\eta) \sim \eta^{-3/2}$. Since the coherent oscillations decrease as $a^{-3}$ there will be a typical curvature scale $H_{\rm c}$ and a typical time $\eta_{\rm c}$ at which the coherent oscillations become dominant with respect to the radiation background. This moment is determined by demanding that

$$H_{\rm r}^2\, M_{\rm P}^2\left(\frac{a_{\rm r}}{a_{\rm c}}\right)^4 \simeq m^2\phi_1^2\left(\frac{a_{\rm m}}{a_{\rm c}}\right)^3, \tag{3.10}$$

which also implies that

$$H_{\rm c} \sim m\varphi^4, \tag{3.11}$$

where $\varphi = \phi_1/M_{\rm P}$. Eq. (3.11) has been obtained without tuning the asymptotic value of $\phi$ to the minimum of its potential. If such a tuning is made, the amplitude of oscillations at $\eta_{\rm m}$ will be smaller than $\phi_1$ and it will be given, according to Eq. (3.9), by $\phi_2(\eta_1/\eta_{\rm r})^{2\alpha-1}$. Thus, the scale $H_c$ will be defined by a different relation namely:

$$m^2\phi_2^2\left(\frac{H_{\rm r}}{H_1}\right)^{\frac{2(2\alpha-1)}{\alpha+1}}\left(\frac{m}{H_{\rm r}}\right)\left(\frac{a_{\rm m}}{a_{\rm c}}\right)^3 \simeq H_{\rm r}^2 M_{\rm P}^2\left(\frac{a_{\rm r}}{a_{\rm c}}\right)^4, \tag{3.12}$$

leading, ultimately, to

$$H_{\rm c} = m\left(\frac{\phi_2}{M_{\rm P}}\right)^4\left(\frac{H_{\rm r}}{H_1}\right)^{\frac{4(2\alpha-1)}{\alpha+1}}\left(\frac{m}{H_{\rm r}}\right)^2. \tag{3.13}$$

In the approximation of instantaneous reheating [i.e. $\eta_{\rm r} \sim \eta_1$], $H_{\rm r} \sim H_1$. Therefore, from Eq. (3.13) $H_{\rm c}$ is smaller than the value determined in Eq. (3.11) by a factor $(m/H_1)^2$. In the approximation of matter-dominated reheating (i.e. $\alpha \sim 2$), the result of instantaneous reheating is further suppressed by a factor $(H_{\rm r}/H_1)^4$ as one can easily argue from Eq. (3.13). From Eqs. (3.5)–(3.6), the evolution of $\phi$ will go as $\eta^{-3}$ when coherent oscillations start dominating.

In spite of the possible tunings made in the asymptotic values of $\phi$, after $\eta_{\rm c}$ there will be a typical time at which the field $\phi$ will decay. In order not to spoil the light elements produced at the epoch of BBN $\phi$ has to decay at a scale larger than $H_{\rm ns} \simeq T_{\rm ns}^2/M_{\rm P}$ (where $T_{\rm ns} \simeq$ MeV. Since $\phi$ is only coupled gravitationally the typical decay scale will be given by comparing the rate with the curvature scale giving that

$$H_{\phi} \sim \Gamma \sim \frac{m^3}{M_{\rm P}^2} > H_{\rm ns}, \tag{3.14}$$

implying that $m > 10^4$ GeV. This requirement also demands that the reheating temperature associated with the decay of $\phi$ will be larger than the BBN temperature.

In order to illustrate some concrete examples of the various possibilities implied by our considerations, suppose that $m \sim 10^3$ TeV and suppose that the asymptotic value of $\phi$ is fine-tuned to its minimum. Furthermore, suppose that the reheating is instantaneous. Then, according to the picture which has been presented, inflation stops at a scale $H_1 \sim 10^{13}$ GeV and $\phi$ starts oscillating at a curvature scale $H_{\rm m} \sim 10^3$ TeV. The coherent oscillations will then become dominant at a curvature scale $H_{\rm c} \sim 10^{-8}$ GeV (having assumed $\phi_2 \sim M_{\rm P}$). The coherent oscillations of $\phi$ will last down to $H_{\phi} \sim 10^{-20}$ GeV. After this moment the Universe will be dominated by the radiation produced in the decay of $\phi$. Notice, for comparison, that the BBN curvature scale is $H_{\rm ns} \simeq 10^{-25}$ GeV so that the decay occurs well before BBN (five orders of magnitude in curvature scale).

Another illustrative example is the one where $m \sim 10^6$ TeV. In this case the decay of $\phi$ occurs prior to the EWPT epoch, namely

$$H_{\phi} > H_{\rm ew}. \tag{3.15}$$

In fact $H_{\rm ew} = \sqrt{N_{\rm eff}}T_{\rm ew}^2/M_{\rm P} \sim 10^{-17}$ GeV (with $N_{\rm eff} = 106.75$ and $T_{\rm ew} \sim 100$ GeV) whereas, from Eq. (3.14), $H_{\phi} \sim 10^{-9}$ GeV. In more general terms we can say that in order to have the $\phi$ decay occurring prior to the EWPT epoch we have to demand that $H_{\phi} > H_{\rm ew}$ which means that $m > 10^5$ TeV.

In closing this section two general comments are in order. If no fine-tuning is made in the asymptotic amplitude of $\phi$, the typical scale of the coherent oscillations will almost coincide with $m$. However, the possibility $\varphi \ll 1$ is still left if, for some reason, we want $H_{\rm c} \ll m$.

The decay of $\phi$ and the consequent freezing of the gauge coupling should occur prior to the EWPT epoch and the baryon number should be generated, in the present context, at the electroweak time. Suppose, for example, that this is not the case and that the BAU has been created prior to the electroweak scale. Suppose, moreover, that the decay of $\phi$ occurs after baryogenesis. Then the temperature of the radiation gas before the decay of $\phi$ will be $T_{\phi} \sim T_{\rm m}(a_{\rm m}/a_{\phi}) \sim m(m/M_{\rm P})^5$. Thus, the entropy increase due to the decay of

$\phi$ will be $\Delta S \sim (T_{\rm decay}/T_{\phi})^3$ where $T_{\rm decay} \sim \sqrt{H_{\phi} M_{\rm P}}$. This implies that $\Delta S \sim m/M_{\rm P}$. It has been observed in different contexts that in order to preserve a pre-existing BAU one should have $\Delta S < 10^5$ [15, 16]. Thus, this bound would imply $m > 10^{14}$ GeV. This is the reason why the present analysis will assume that the decay of $\phi$ occurs prior to the electroweak time and that the BAU is generated at the EWPT or shortly after.

## 3.2 Evolution of the inhomogeneous modes

When the Universe passes from the inflationary stage to the subsequent radiation dominated expansion, inhomogeneities of the field $\phi$ are generated. This may invalidate the original assumptions and introduce further complications by adding qualitatively new constraints on the scenario.

It is useful to recall that the inhomogeneities of $\phi$ can be interpreted, in the framework of second quantization, as quanta of the field $\phi$. Hence, the inhomogeneities produced because of the sudden change of the geometry from the de Sitter epoch to the radiation dominated epoch can be counted by estimating the number of quanta produced by the sudden change of the geometry according to the well known techniques of curved space-times [17].

Consider the first order fluctuations of the field $\phi$

$$\phi(\vec{x},\eta) = \phi(\eta) + \delta\phi(\vec{x},\eta), \tag{3.16}$$

whose evolution equation is, in Fourier space,

$$\psi'' + 2\mathcal{H}\psi' + [k^2 + m^2 a^2]\psi = 0, \tag{3.17}$$

where $\psi(k,\eta)$ is the Fourier component of $\delta\phi(\vec{x},\eta)$. In order to count the number of quanta produced during the transition of the geometry from the inflationary to the radiation dominated stage of expansion the (canonically normalized) amplitude of fluctuations $\Psi = \psi a$ should be defined so that Eq. (3.17) becomes:

$$\Psi'' + [k^2 + m^2 a^2 - \frac{a''}{a}]\Psi = 0. \tag{3.18}$$

In the de Sitter stage of expansion Eq. (3.18) reduces to

$$\Psi_{\rm i}'' + [k^2 + \frac{\mu^2 - 2}{\eta^2}]\Psi_{\rm i} = 0, \tag{3.19}$$

whereas during the radiation dominated stage of expansion Eq. (3.19) takes the form

$$\Psi_{\rm r}'' + [k^2 + \frac{\mu^2(\eta + 2\eta_1)^2}{\eta_1^4}]\Psi_{\rm r} = 0. \tag{3.20}$$

The solution of Eq. (3.19) (with the correct quantum-mechanical normalization for $\eta \to -\infty$) can be written as

$$\Psi_{\rm i}(\eta) = \frac{1}{\sqrt{2k}}\; p\; \sqrt{-x}\; H_{\rho}^{(1)}(-x), \tag{3.21}$$

where $x = k\eta$ and $H_{\nu}^{(1)}$ is the first order Hankel function [13]. In the pure de Sitter case, $\rho = 3/2\sqrt{1-(4/9)\mu^2}$ and since $\mu \ll 1$, $\rho \simeq 3/2$; $p$ is a phase factor which has been chosen in such a way that

$$p = \sqrt{\frac{\pi}{2}}\; e^{i\frac{\pi}{4}(1+2\rho)}. \tag{3.22}$$

With this choice of $p$ we have that $\Psi_{\rm i}(\eta) \sim e^{-ik\eta}/\sqrt{2k}$ for $\eta \to -\infty$.

During the radiation dominated stage of expansion Eq. (3.20) is the equation of parabolic cylinder functions [13]. The solutions turning into positive and negative frequencies for $\eta \to +\infty$ are then

$$\begin{aligned} f_{\rm r}(\eta) &= \frac{1}{(2\gamma)^{1/4}}\, e^{i\frac{\pi}{8}} D_{-iq-\frac{1}{2}}(ie^{-i\frac{\pi}{4}}z), \\ f_{\rm r}^{*}(\eta) &= \frac{1}{(2\gamma)^{1/4}}\, e^{-i\frac{\pi}{8}} D_{iq-\frac{1}{2}}(e^{-i\frac{\pi}{4}}z), \end{aligned} \tag{3.23}$$

where

$$z = \sqrt{2\gamma}(\eta + 2\eta_1), \quad q = \frac{k^2}{2\gamma}, \tag{3.24}$$

and where $D_{\sigma}$ are the parabolic cylinder functions in the Whittaker's notation. The solution of Eq. (3.20)

$$\Psi_{\rm r}(\eta) = c_{+}(k) g_{\rm r}(\eta) + c_{-}(k) g_{\rm r}^{*}(\eta), \tag{3.25}$$

is given in terms of $c_{+}(k)$ and $c_{-}(k)$ which are the two (complex) Bogoliubov coefficients satisfying $|c_{+}(k)|^2 - |c_{-}|^2 = 1$. In a second quantized approach $|c_{-}(k)|^2$ is the mean number of created quanta, whereas in a semi-classical approach $c_{-}(k)$ can be viewed as the coefficient parametrising the mixing between positive and negative frequency modes. In the case $c_{-}(k) \simeq 0$ no mixing takes place and no amplification is produced. In order to determine $c_{\pm}(k)$, $\Psi_{\rm i}(\eta)$ and $\Psi_{\rm r}(\eta)$ should be continuously matched in $\eta = -\eta_1$, namely

$$\begin{aligned} \Psi_{\rm i}(-\eta_1) &= \Psi_{\rm r}(-\eta_1), \\ \Psi_{\rm i}'(-\eta_1) &= \Psi_{\rm r}'(-\eta_1), \end{aligned} \tag{3.26}$$

By solving this system, an exact expression for the Bogoliubov coefficients is obtained which is, in general a function of two variables : $\mu = m\eta_1$ and $x_1 = k\eta_1$. Since $\mu \ll 1$ the exact result can be expanded, in this limit,

$$\begin{aligned} c_{+}(k) &= \pi e^{i\frac{\pi}{8}}\Big\{\frac{i}{\sqrt{2}\Gamma(\frac{3}{4})} S_2(x_1,\rho)\mu^{-\frac{1}{4}} + \frac{(1+i)}{2\Gamma(\frac{1}{4})}[S_1(x_1,\rho) + S_2(x_1,\rho)]\mu^{\frac{1}{4}}\Big\} + \mathcal{O}(\mu^{\frac{5}{4}}), \\ c_{-}(k) &= \pi e^{-i\frac{\pi}{8}}\Big\{-\frac{i}{\sqrt{2}\Gamma(\frac{3}{4})} S_2(x_1,\rho)\mu^{-\frac{1}{4}} + \frac{(i-1)}{2\Gamma(\frac{1}{4})}[S_1(x_1,\rho) + S_2(x_1,\rho)]\mu^{\frac{1}{4}}\Big\} + \mathcal{O}(\mu^{\frac{5}{4}}), \end{aligned} \tag{3.27}$$

where $S_1(x_1,\rho)$ and $S_2(x_1,\rho)$ contain the explicit dependence upon the Hankel's functions:

$$S_1(x_1,\rho) = e^{i\frac{\pi}{4}(1+2\rho)} H^{(1)}_{\rho}(x_1),$$
$$S_2(x_1,\rho) = \sqrt{x_1} e^{i\frac{\pi}{4}(1+2\rho)} \Big[(\rho+\frac{1}{2})\frac{H^{(1)}_{\rho}(x_1)}{\sqrt{x_1}} - \sqrt{x_1} H^{(1)}_{\rho+1}(x_1)\Big]. \qquad (3.28)$$

If $\rho \sim 3/2$ Eq. (3.28) gives

$$c_{+}(k) = e^{i\frac{\pi}{8}}\sqrt{\pi}\Big\{-\frac{x_1^{-\frac{3}{2}}}{2\Gamma(\frac{3}{4})\mu^{\frac{1}{4}}} + \frac{i\,x_1^{-\frac{1}{2}}}{2\Gamma(\frac{3}{4})\mu^{\frac{1}{4}}} + \Big[\frac{1}{2\Gamma(\frac{3}{4})\mu^{\frac{1}{4}}} + \frac{(i-1)\mu^{1/4}}{\sqrt{2}\Gamma(\frac{1}{4})}\Big]\sqrt{x_1}\Big\} + \mathcal{O}(\mu^{\frac{5}{4}}),$$
$$c_{-}(k) = e^{-i\frac{\pi}{8}}\sqrt{\pi}\Big\{\frac{x_1^{-\frac{3}{2}}}{2\Gamma(\frac{3}{4})\mu^{\frac{1}{4}}} - \frac{i\,x_1^{-\frac{1}{2}}}{2\Gamma(\frac{3}{4})\mu^{\frac{1}{4}}} + \Big[-\frac{1}{2\Gamma(\frac{3}{4})\mu^{\frac{1}{4}}} + \frac{(i+1)\mu^{1/4}}{\sqrt{2}\Gamma(\frac{1}{4})}\Big]\sqrt{x_1}\Big\} + \mathcal{O}(\mu^{\frac{5}{4}}) \qquad (3.29)$$

In the limit $x_1 = k\eta_1 \ll 1$ the mean number of created quanta can be finally approximated as

$$\overline{n}(k) \simeq |c_{-}(k)|^2 = q|k\eta_1|^{-2\rho}\mu^{-1/2} \qquad (3.30)$$

where $q$ is a numerical coefficient of the order of $10^{-2}$. The energy density of the created (massive) quanta can be estimated from

$$d\rho_{\psi} = \frac{d^3\omega}{(2\pi)^3}\, m\, \overline{n}(k) \qquad (3.31)$$

where $\omega = k/a$ is the physical momentum. In the case of a de Sitter phase ($\rho = 3/2$) the typical energy density of the produced fluctuations is

$$\rho_{\psi}(\eta) \simeq q\, m\, H_1^3 \left(\frac{m}{H_1}\right)^{-1/2}\left(\frac{a_1}{a}\right)^3 \qquad (3.32)$$

The produced massive quanta may become dominant. If they become dominant after $\phi$ already decayed they will not lead to further constraints on the scenario. If they become dominant prior to the decay of $\phi$ further constraints may be envisaged. The scale at which the massive fluctuations become dominant with respect to the radiation background can be determined by requiring that $\rho_{\psi}(\eta_*) \simeq \rho_{\gamma}(\eta_*)$ implying that

$$q\, m\, H_1^3\left(\frac{m}{H_1}\right)^{-1/2}\left(\frac{a_1}{a_*}\right)^3 \simeq H_1^2\, M_{\rm P}^2\left(\frac{a_1}{a_*}\right)^4, \qquad (3.33)$$

which translates into

$$H_* \simeq q^2\, m\epsilon^4, \qquad (3.34)$$

where $\epsilon = H_1/M_{\rm P}$. In order to make sure that the non-relativistic modes will become dominant after $\phi$ already decayed $H_* < H_{\phi}$ should be imposed, that is to say $m > 10^2$ TeV for $\epsilon \sim 10^{-6}$.

The maximum tolerable amount of entropy, in order not to wash-out any preexisting BAU is model-dependent but, in general, $\Delta S < 10^5$ seems to be acceptable [14, 15, 16]. Defining $T_\phi$ as the radiation gas already present at the scale $H_\phi$, the entropy increase from $T_\phi$ to $T_{\rm decay} \simeq \sqrt{H_\phi M_{\rm P}}$ is of the order of

$$\Delta S = \left(\frac{T_{\rm decay}}{T_\phi}\right)^3, \tag{3.35}$$

where

$$T_\phi = T_*\left(\frac{a_*}{a_\phi}\right) \simeq m\ \xi^{1/6}\ \epsilon^{-1/2}, \tag{3.36}$$

where $\xi = m/M_{\rm P}$. Demanding that $\Delta S < 10^5$ implies that

$$\xi > 10^{-10}\epsilon^3. \tag{3.37}$$

Taking, as usual, $\epsilon \simeq 10^{-6}$, $m > 10$ GeV.

Hence, if the constraints pertaining to the homogeneous mode are enforced, the bounds coming from the inhomogeneous modes do not invalidate the conclusions of the analysis. According to the logic expressed in Eq. (3.15), an illustrative example is the case where $m > 10^5$ TeV and the BAU is generated after EWPT. In this case the bounds obtained in the present section are satisfied and the analysis of the evolution of the inhomogeneous modes shows that the qualitatively new bounds introduced in the picture are less constraining than the ones obtained in the analysis of the dynamics of the homogeneous mode.

# 4 Evolution of the gauge field fluctuations

The evolution of the field $\phi$ during and after the de Sitter stage implies, according to Eqs. (3.3)–(3.9), that the two-point function of the gauge field fluctuations may very well grow.

For $\eta < -\eta_1$, a rough approximation suggests that the effect of the ohmic current is not present and the gauge field is in the vacuum state with $k/2$ energy in each of its modes. By promoting the classical fields to quantum mechanical operators we have that the physical polarizations of the magnetic field can be written as

$$\hat{b}_i(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_\alpha e_i^\alpha [\hat{a}_{k,\alpha} b(k\eta) e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger_{-k,\alpha} b^*(k\eta) e^{-i\vec{k}\cdot\vec{x}}], \tag{4.1}$$

where $b(k\eta) = B(k\eta)/g(\eta)$ obey the equation

$$b'' + \left[k^2 - 2\left(\frac{g'}{g}\right)^2 + \frac{g''}{g}\right] b = 0. \tag{4.2}$$

Notice that $b(k\eta)$ are the correct normal modes whose limit (for $\eta \to -\infty$) should be normalized to $\sqrt{k/2}e^{-ik\eta}$. The two-point correlation function of the magnetic fluctuations can then be expressed as

$$\mathcal{G}_{ij}(\vec{r},\eta) \equiv \langle \hat{b}_i(\vec{x},\eta)\hat{b}_j(\vec{x}+\vec{r},\eta)\rangle = \int \frac{d^3k}{(2\pi)^3} P_{ij}(k) b(k,\eta) b^*(k,\eta) e^{i\vec{k}\cdot\vec{r}}, \tag{4.3}$$

where

$$P_{ij}(k) = (\delta_{ij} - \frac{k_i k_j}{k^2}). \tag{4.4}$$

The magnetic energy density, derived from the energy-momentum tensor corresponding to the action of Eq. (2.2), is related to the trace of the correlation function reported in Eq. (4.1) over the physical polarizations:

$$\rho_B(r,\eta) = \int \rho_B(k,\eta) \frac{\sin kr}{kr} \frac{dk}{k} \tag{4.5}$$

where

$$\rho_B(k,\eta) = \frac{1}{\pi^2} k^3 |b(k,\eta)|^2. \tag{4.6}$$

A necessary condition in order to assess that gauge field fluctuations grow during the de Sitter phase is that the two-point function increases in the limit $\eta \to -\eta_1$ [18, 19].

Suppose that the gauge coupling decreases with monotonic dependence upon the field $\phi$, namely

$$g(\eta) = (\frac{\phi - \phi_1}{M_{\rm P}})^{\frac{\lambda}{2}}, \ \ \lambda > 0. \tag{4.7}$$

This parametrisation is purely phenomenological, however, it allows to take into account, at once, some physically interesting cases like the one suggested by the low-energy string effective action where, in the limit of $\phi/M_{\rm P} < 1$, $g^2(\phi) \sim \phi$.

Using Eq. (3.3) and Eq. (4.7) into Eq. (4.2) the time evolution of the normal modes of the magnetic field can be found analytically since the specific form of Eq. (4.2) falls in the same category of Eq. (3.19). Hence,

$$b(k,\eta) = N\sqrt{k\eta} H_\nu^{(2)}(k\eta), \ \ N = \frac{\sqrt{k\pi}}{2} e^{-i\frac{\pi}{4}(1+2\nu)}, \tag{4.8}$$

with $\nu = (3\lambda+1)/2$ and where $H_\nu^{(2)}(k\eta)$ the Hankel function of second kind [13]. Notice that the normalization $N$ has been chosen in such a way that for $\eta \to -\infty$ the correct quantum mechanical normalization is reproduced. Consequently, following Eq. (4.3), the two-point function evolves as

$$\lim_{\eta \to -\eta_1} \mathcal{G}_{ij}(r,\eta) \sim \left|\frac{\eta}{\eta_1}\right|^{-3\lambda}. \tag{4.9}$$

Since the two-point function increases magnetic fluctuations are generated.

For sake of completeness the case of increasing gauge coupling will now be examined using the following phenomenological parameterisation

$$g(\eta) = (\frac{\phi - \phi_1}{M_{\rm P}})^{-\frac{\delta}{2}}, \ \ \delta > 0. \tag{4.10}$$

Again different scenarios can be imagined. For instance, one coould argue in favour of scenarios where the gauge coupling depends upon $\phi$ (or upon $\eta$) in a highly non-monotonic way. For the illustrative purposes of the present investigation it is however sufficient to focus the attention on the case of monotonic dependence.

Following now the same steps outlined in the case of decreasing gauge coupling the evolution of the two-point function can be obtained

$$\lim_{\eta \to -\eta_1} \mathcal{G}_{ij}(r, \eta) \sim \left|\frac{\eta}{\eta_1}\right|^{2-3\delta}. \tag{4.11}$$

for $\delta > 1/3$ and

$$\lim_{\eta \to -\eta_1} \mathcal{G}_{ij}(r, \eta) \sim \left|\frac{\eta}{\eta_1}\right|^{3\delta}. \tag{4.12}$$

for $\delta < 1/3$. If $\delta < 1/3$ the correlation function decreases and this signals that large scale magnetic fields are not produced.

The back-reaction of the produced fluctuations can be safely neglected in de Sitter space. Looking at Eqs. (2.3) and (2.11) it can happen that if the magnetic fluctuations grow too much the term at the right hand side will become of the same order of the others. This is not the case. Using the conventions of this Section together with the explicit form of the scale factor in the de Sitter phase it can be shown that

$$\frac{1}{g^3 a^2}\frac{\partial g}{\partial \phi}\vec{B}^2 \sim \left|\frac{\eta}{\eta_1}\right|^{-2\nu} |k\eta_1|^{5-2\nu} \tag{4.13}$$

where $\nu$ is determined from the specific power dependence of the coupling as a function of $\phi$. Since we are interested in large scale modes we have $k\eta_1 \ll 1$. Therefore the back reaction effects are relevant towards the end of the de Sitter phase (i.e. $\eta \sim -\eta_1$ ) and for $k \sim \eta_1^{-1}$, namely exactly for the modes not relevant for the present investigation.

After the end of inflation the onset of the conductivity dominated regime may not be instantaneous. In this case after $\eta_1$ the presence of a reheating phase should be taken into account. Suppose, for instance, that $g(\eta)$ decreases according to Eq. (4.7). Suppose, moreover, that during reheating the background is dominated by the coherent oscillations of the inflaton. In this case the effective evolution of the geometry will be dominated by matter with $a(\eta) \sim \eta^2$. According to Eq. (3.4) (with $\alpha \sim 2$), $\phi \sim \eta^{-3}$. The value of the magnetic inhomogeneities at $\eta_{\rm r}$ will be given by solving Eq. (4.2)

$$b(k, \eta_{\rm r}) \sim A_1\left(\frac{\eta_{\rm r}}{\eta_1}\right)^{\frac{3}{2}\lambda} + A_2\left(\frac{\eta_{\rm r}}{\eta_1}\right)^{1-\frac{3}{2}\lambda}, \tag{4.14}$$

where $A_1$ and $A_2$ are two arbitrary constants. Depending upon the value of $\lambda$ the fastest growing solution is selected in Eq. (4.14). This phase may induce further amplification on the two-point function since its main effect is to delay the conductivity-dominated regime.

## 4.1 Generalized MHD equations

For $\eta > \eta_r$ the role of the Ohmic diffusion becomes important and the evolution of the magnetic inhomogeneities will be described by the MHD equations generalized to the case of time varying gauge coupling.

The ordinary (i.e. fixed coupling) MHD treatment is an effective description valid for length scales larger than the Debye radius and for frequencies smaller than the iono-acoustic frequency. This means that MHD is accurate in reproducing the spectrum of plasma excitations obtained from the full kinetic (Vlasov-Landau) approach but only for sufficiently low frequencies and for sufficiently large scales.

Implicit in the ordinary MHD analysis is the assumption that the plasma has to be electrically neutral ($\vec{\nabla} \cdot \vec{E} = 0$) over length scales larger than the Debye radius. Thus, this system of equations cannot be applied for distances shorter than the Debye radius and for frequencies larger than the plasma frequency where a kinetic description should be employed.

MHD equations can be derived from a microscopic (kinetic) approach and also from a macroscopic approach where the displacement current is neglected [20]. If the displacement current is neglected the electric field can be expressed using the Ohm law and the magnetic diffusivity equation can be derived

$$\frac{\partial \vec{B}}{\partial \eta} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \frac{1}{\sigma} \nabla^2 \vec{B}. \tag{4.15}$$

The term containing the bulk velocity field is called dynamo term and it receives contribution provided parity is globally broken over the physical size of the plasma. In Eq. (4.15) the contribution containing the conductivity is usually called magnetic diffusivity term.

In the superconducting (or ideal) approximation the resistivity of the plasma goes to zero and the induced (Ohmic) electric field is orthogonal both to the bulk velocity of the plasma and to the magnetic field [i. e. $\vec{E} \simeq -\vec{v} \times \vec{B}$]. In the real (or resistive) approximation the resistivity may be very small but it is always finite and the Ohmic field can be expressed as

$$\vec{E} \simeq \frac{\vec{\nabla} \times \vec{B}}{\sigma} - \vec{v} \times \vec{B}. \tag{4.16}$$

If the gauge coupling changes with time the system of equations obtained by neglecting the displacement current receives new contributions and the relevant equations can be obtained, in the resistive approximation:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial \eta}, \tag{4.17}$$

$$\vec{E} = \frac{\vec{J}}{\sigma} - \vec{v} \times \vec{B}, \tag{4.18}$$

$$\frac{1}{g^2}\vec{\nabla} \times \vec{B} = \vec{J} - 2\frac{g'}{2g^3}\vec{E}, \tag{4.19}$$

Using Eqs. (4.17)–(4.19) the generalized magnetic diffusivity equation can be obtained:

$$(1 - \frac{2}{\sigma\ g^2}\frac{g'}{g})\frac{\partial \vec{B}}{\partial \eta} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \frac{1}{\sigma g^2}\nabla^2 \vec{B}. \tag{4.20}$$

Notice that Eq. (4.20) reproduces Eq. (4.15) if $g' \to 0$.

Suppose now that the plasma, whose effective Ohmic description has been presented, is relativistic. In the case when the coupling is constant $\sigma$ is constant and it is given by

$$\sigma \equiv \sigma_c(\eta) a(\eta), \tag{4.21}$$

where $\sigma_c \sim T/g^2$ scales as the inverse of $a(\eta)$ if the evolution of the Universe is, to a good approximation, adiabatic.

If $g$ is not constant, $\sigma$ is not constant anymore but it decreases if the gauge coupling increases and, vice versa, it increases if the gauge coupling decreases. In spite of this, in the generalized MHD equations, the combination which appears is always $\sigma g^2$ which is roughly constant for an adiabatically expanding Universe.

In the approximation of instantaneous reheating, the solution of Eq. (4.20) is given by

$$B(k, \eta) = B(k, \eta_1) e^{-\int \frac{k^2}{\sigma g^2 - 2\frac{g'}{g}} d\eta}. \tag{4.22}$$

According to Eqs. (3.13)–(3.14), in order to get the coupling frozen prior to $H_{\rm ew} \sim 10^{-17}$ GeV, $m > 10^5$ TeV shall be required. If the gauge coupling is always decreasing as a function of $\eta$, it can be parametrized by Eq. (4.7). Hence Eq. (4.22) can be evaluated by using the explicit evolution of $\phi$ as obtained from Eqs. (3.8)–(3.9) implying that $\phi_{\rm r} \sim \eta^{-1}$, $\phi_{\rm m} \sim \eta^{-3/2}$ and $\phi_{\rm c} \sim \eta^{-3}$. The result is

$$B(k, \eta_0) = \mathcal{I}(\eta_1, \eta_{\rm m}, \eta_\phi, \eta_{\rm c}) e^{-\frac{k^2}{\sigma g^2}(\eta_1 + \eta_0)} \tag{4.23}$$

where $\eta_0$ is the present time and

$$\mathcal{I}(\eta_1, \eta_{\rm m}, \eta_\phi, \eta_{\rm c}) = [(\frac{\lambda + \sigma g^2 \eta_1}{\lambda + \sigma g^2 \eta_{\rm m}})(\frac{3\lambda + 2\sigma g^2 \eta_{\rm m}}{3\lambda + 2\sigma g^2 \eta_{\rm c}})^{\frac{3}{2}}(\frac{3\lambda + \sigma g^2 \eta_{\rm c}}{3\lambda + \sigma g^2 \eta_\phi})^3]^{-\lambda[\frac{k}{\sigma g^2}]^2}. \tag{4.24}$$

Concerning Eqs. (4.23)–(4.24) few comments are in order. From Eq. (4.23) all the modes

$$k^2 > k_\sigma^2 \sim \frac{\sigma g^2}{\eta_0} \tag{4.25}$$

are suppressed by the effect of the conductivity. The present value of $\omega_\sigma(\eta_0)$ [4] can be estimated by recalling that $1/\eta_0 \sim H_0 a_0$ where $H_0 \sim 10^{-61}\,M_{\rm P}$. Thus $\omega_\sigma \sim 10^{-3}$ Hz. Present modes of the magnetic fields are dissipated if $\omega > \omega_\sigma$.

As far as the problem of galactic magnetic fields is concerned, the relevant set of scales range around the Mpc corresponding to present modes of the magnetic field $\omega_{\rm G} \sim 10^{-14}$ Hz, i.e. $\omega_{\rm G} \ll \omega_\sigma$.

# 5 Estimates of large scale magnetic fields

In the approximation of instantaneous reheating the typical (present) frequency corresponding to the end of inflation can be computed and it turns out to be

$$\omega_1(\eta_0) \sim z_{\rm dec}^{-1}\, T_{\rm dec}\, \epsilon^{1/2}\, \xi^{1/3} \varphi^{-\frac{2}{3}}. \tag{5.1}$$

Since $T_{\rm dec} \sim 0.26$ eV , $z_{\rm dec}^{-1} T_{\rm dec} \sim 100\,{\rm GHz}$. Eq. (5.1) can be obtained by red-shifting the highest mode, i.e. $\omega_1(\eta_1) \sim H_1$ through the different stages of the evolution of the model, namely, according to Eqs. (3.11) and (3.14), from $\eta_1$ down to $\eta_{\rm m}$ and from $\eta_{\rm m}$ down to $\eta_{rmc}$. Recall that from $\eta_{\rm c}$ to $\eta_\phi$ the Universe is, effectively, matter dominated. The other typical frequencies appearing in the time evolution of the gauge coupling can be written, in units of $\omega_1(\eta_0)$, as

$$\begin{aligned}
&\frac{\omega_{\rm m}(\eta_0)}{\omega_1(\eta_0)} = \epsilon^{-1/2}\ \xi^{-1/2},\\
&\frac{\omega_{\rm c}(\eta_0)}{\omega_1(\eta_0)} = \xi^{1/2}\ \epsilon^{-1/2}\ \varphi,\\
&\frac{\omega_{\phi}(\eta_0)}{\omega_1(\eta_0)} = \epsilon^{-1/2}\ \xi^{7/6}\ \varphi^{2/3},
\end{aligned} \tag{5.2}$$

where, as in the case of Eq. (5.1) all the frequencies are evaluated at the present time.

In the case of decreasing gauge coupling [described by Eq. (4.7)] the amount of generated large scale magnetic field can be estimated from Eqs. (4.2)–(4.8) together with Eqs. (4.20)–(4.23). Bearing in mind that the typical frequency scale corresponding to 1 Mpc is $10^{-14}$ Hz we have that the ratio of the magnetic to radiation energy density is:

$$r_B(\omega_{\rm G}) = f(\lambda)\ \epsilon^{\frac{3}{2}\lambda}\ \xi^{\lambda-\frac{4}{3}}\ \varphi^{2\lambda-\frac{8}{3}}\ 10^{-25(4-3\lambda)}\ \mathcal{T}(\omega_{\rm G}), \tag{5.3}$$

[4] With $\omega(\eta) \sim k/a(\eta)$ the physical momentum will be denoted.

where

$$f(\lambda) = \frac{2^{3\lambda-1}}{\pi^3}\Gamma^2\Big(\frac{3}{2}\lambda + \frac{1}{2}\Big), \tag{5.4}$$

and

$$\mathcal{T}(\omega_{\rm G}) \simeq e^{-\frac{\omega_{\rm G}^2}{\omega_\sigma^2}}[(\frac{\omega_\phi}{\omega_1})(\frac{\omega_\phi}{\omega_{\rm m}})^{\frac{1}{2}}(\frac{\omega_\phi}{\omega_{\rm c}})^{\frac{3}{2}}]^{-\lambda\frac{\omega_{\rm G}^2}{T_0^2}}, \tag{5.5}$$

which means, using Eqs. (5.1)–(5.2),

$$\mathcal{T}(\omega_{\rm G}) = e^{-\frac{\omega_{\rm G}^2}{\omega_\sigma^2}}[\epsilon^{-1/2}\varphi^{-1}\xi^{5/2}]^{-\lambda\frac{\omega_{\rm G}^2}{T_0^2}}. \tag{5.6}$$

Notice that in Eq. (5.5) is the present CMB temperature.

In order to illustrate the regions of the parameter space where magnetogenesis is possible $\varphi \sim 1$ will be assumed. In Fig. 1 and Fig. 2 the case of decreasing gauge coupling is discussed. In Fig. 1, $\lambda$ is fixed and the exclusion plot is given in terms of $\epsilon$ and $\xi$. In particular, for illustration, $\lambda = 1$ has been chosen. The choice of $\lambda \sim 1$ implies that $g^2(\phi) \sim \phi$. Such a case is favoured from the tree-level string effective action where the effective coupling can be approximated as $g^2(\phi) \sim \phi/M_{\rm P}$ for $\phi < M_{\rm P}$.

The two vertical lines lines mark, respectively, the bounds coming from BBN [i.e. $\xi > 10^{-15}$, from Eq. (3.14)] and from the electroweak epoch [i.e. $\xi > 10^{-11}$, from Eq. (3.15)]. For consistency with the assumptions of Eqs. (3.2)–(3.3) $m/H_1 \ll 1$, implying $\epsilon \gg \xi$. With the lower dashed line the bound $\epsilon > \xi$ is reported. Notice that the requirement of Eq. (3.37) is not numerically relevant. The upper dashed line is obtained by requiring, according to Eq. (3.34), that $H_\phi > H_*$. Finally, the full (diagonal) line is derived by imposing on Eqs. (5.3)–(5.6) the dynamo requirement.

In Fig. 2 the value of $\xi$ has been fixed to $10^{-11}$, as required by the considerations related to the electroweak epoch and the magnetogenesis region is described in terms of $\epsilon$ and $\lambda$, both varying over their physical range. The two horizontal lines fix the bounds coming from $\epsilon \leq 10^{-6}$ and from $\epsilon > \xi$. With the dashed line the curve $r_B(\omega_{\rm G}) \sim 10^{-20}$ is denoted. Hence, values larger than the dynamo requirement are allowed. This observation may be relevant in the context of magnetic fields associated with clusters. In Fig. 2, $\lambda$ lies in the range $0 < \lambda < 4/3$. This choice guarantees the growth of the correlation function of the magnetic inhomogeneities during the de Sitter stage.

In order not to conflict with large scale bounds coming from the isotropy of the CMB the energy spectra of the produced gauge field fluctuations have to decay at large distance scales, implying that $0 < \lambda \leq 4/3$. To be compatible with CMB anisotropies $r_{\rm B}(\omega_{\rm dec}) \leq 10^{-10}$ should be imposed (where $\omega_{\rm dec} \simeq 10^{-16}$ Hz). If the condition $0 < \lambda < 4/3$ is enforced, the spectra increase with frequency and the possible bounds coming from the anisotropy of the CMB are satisfied.

An analogous estimate can be obtained in the case of increasing gauge coupling discussed in Eq. (4.10). In this case

$$r_{\rm B}(\omega_{\rm G}) \sim f(\delta)\epsilon^{\frac{3}{2}\delta-1}\ \xi^{\delta-2}\ \varphi^{4-2\delta}\ 10^{-25(6-3\delta)}\mathcal{T}(\omega_{\rm G}), \tag{5.7}$$

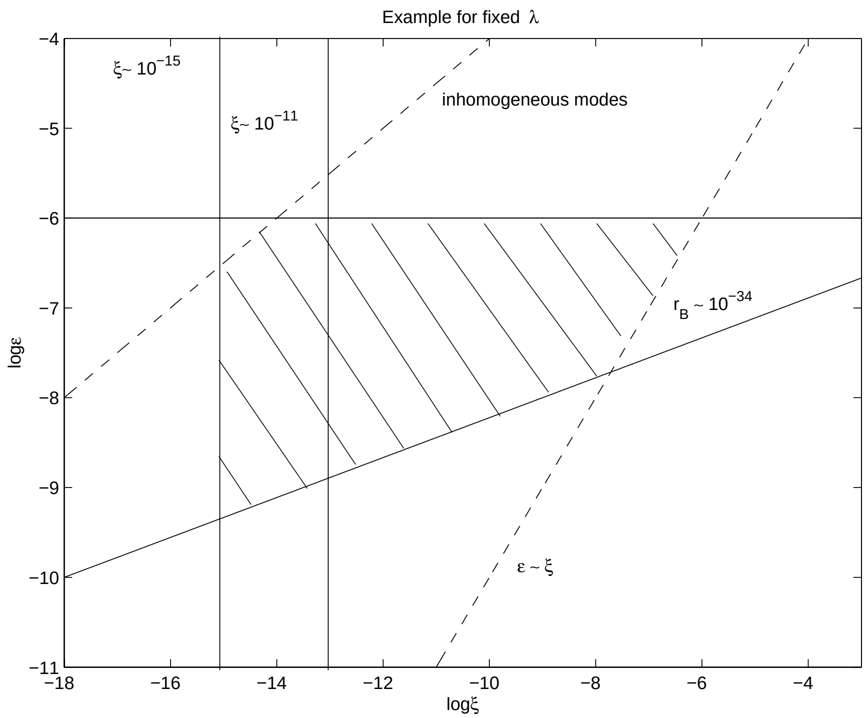


Figure 1: The shaded area illustrates the region where magnetogenesis is possible in the case where $\lambda = 1$ and $\varphi \sim 1$. The vertical lines correspond to the requirements coming from BBN and from the EWPT epoch.

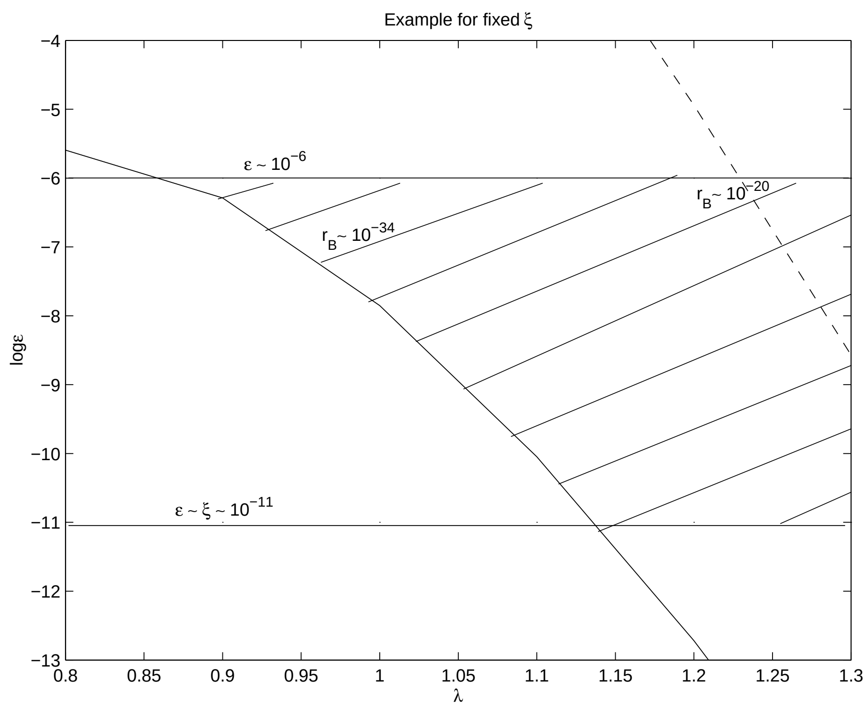


Figure 2: The region with stripes defines the area where magnetogenesis can occur in the case of fixed mass [i.e. $\xi \sim 10^{-11}$] and for $\varphi \sim 1$.

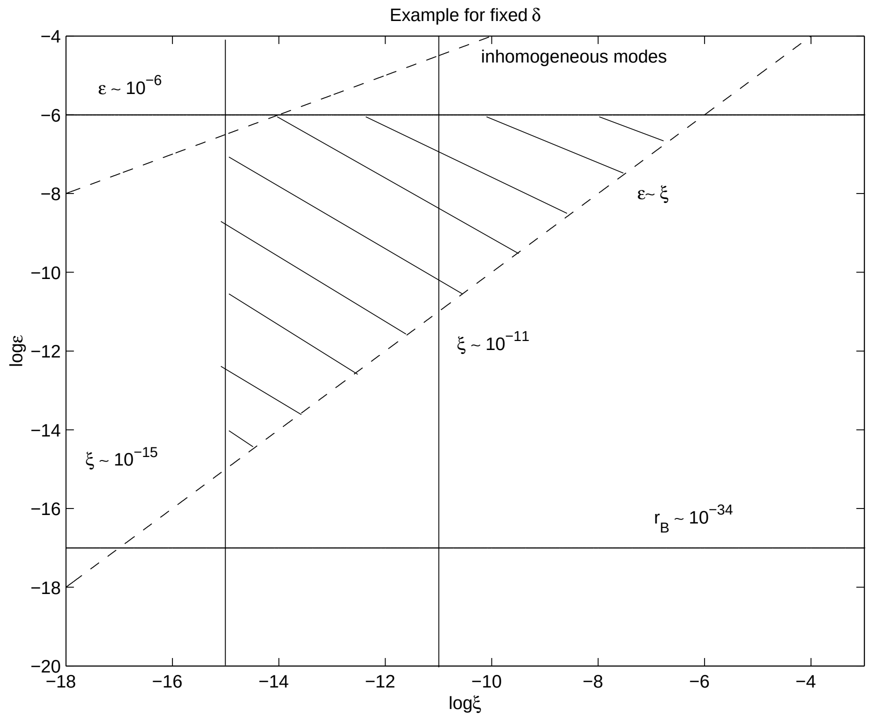


Figure 3: The magnetogenesis region in the case $\delta = 2$ (i.e. increasing gauge coupling) and $\varphi \sim 1$. The exclusion plot is then given in terms of $\epsilon$ and $\xi$.

where

$$f(\delta) = \frac{2^{3\delta-3}}{\pi^3}\Gamma^2\biggl(\frac{3}{2}\delta - \frac{1}{2}\biggr), \tag{5.8}$$

and

$$\mathcal{T}(\omega_{\rm G}) = e^{-\frac{\omega_{\rm G}^2}{\omega_\sigma^2}}\bigl[\epsilon^{-1/2}\varphi^{-1}\xi^{5/2}\bigr]^{\delta\frac{\omega_{\rm G}^2}{T_0^2}}. \tag{5.9}$$

In Fig. 3 and Fig. 4 the requirements coming from the dynamo mechanism as well as the other theoretical constraints are illustrated. For both plots $\epsilon < 10^{-6}$ and $\varphi \sim 1$. In Fig. 3 the case $\delta = 2$ is illustrated. The shaded area selects the region of the parameter space where the dynamo requirement is imposed on Eq. (5.7). The two vertical lines illustrate the conditions of Eqs. (3.14)–(3.15). As in Fig. 2 the two dashed lines correspond to the constraints coming from the inhomogeneous modes and from the condition $\epsilon > \xi$.

The requirement that the spectra decrease at large distance scales implies, in this case that $\delta \leq 2$. The condition on the growth of the correlation function obtained in Eqs. (4.11)–(4.12) imply $\delta > 1/3$. Thus the interesting physical range of Eq. (4.10) and (5.7) will be $1/3 < \delta \leq 2$.

In Fig. 4 the parameter space is illustrated for fixed values of $\xi$, i.e. $\xi \sim 10^{-11}$. As in

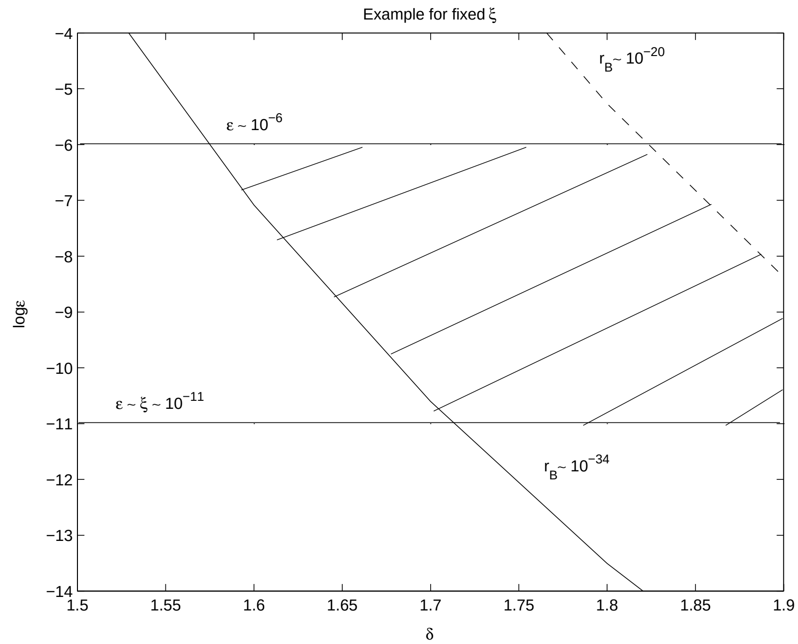


Figure 4: The magnetogenesis region is illustrated in the case of increasing gauge coupling in the ($\epsilon$, $\delta$) plane. Notice that in the present example $\xi \sim 10^{-11}$ (for compatibility with the EW epoch) and $\varphi \sim 1$.

Fig. 3 the full and dashed curves correspond to the dynamo requirements imposed on Eq. (5.7). The shaded area selects the allowed region in the space of the parameters where magnetogenesis is possible for $10^{-11} < \epsilon < 10^{-6}$. As in the case of Fig. 2 there are regions in the shaded area where values much larger than the magnetogenesis requirement are possible (see the dashed line in Fig. 4).

As it has been pointed out in deriving the theoretical bounds on the scenario the requirement $\xi \geq 10^{-11}$ may be too restrictive since it excludes the variation of the gauge coupling at the electroweak time. To relax this assumption is possible and it would require a precise analysis of the dynamics of the EWPT in the presence of time varying gauge coupling. At the moment this kind of analysis is not available. Summarizing this illustrative discussion, there are regions in the parameter space of the models wher all the theoretical constraints are satisfied and where magnetogenesis is possible. In particular, it is possible that the gauge coupling freezes prior to the electroweak epoch, leading still to magnetogenesis.

# 6 Concluding remarks

There are no reasons why the gauge couplings should be constant throughout all the history of the Universe. If they are allowed to change prior to the formation of the light elements they can lead to computable differences in the cosmological evolution.

In the present paper the interplay between inflationary magnetogenesis and the evolution of the Abelian gauge coupling has been addressed. In a phenomenologically reasonable model of inflationary and post-inflationary evolution the relaxation of the gauge coupling leads to a growth in the correlation function of magnetic inhomogeneities. Large scale magnetic fields are then generated. The evolution of the gauge coupling is driven, in the present context, by a massive scalar.

Since the gauge coupling evolves significant changes in the evolution of the plasma can be envisaged. In the present investigation the ordinary MHD equations have been generalized to the case of time evolving gauge coupling but other effects could be envisaged. In particular, the generlization of the full kinetic approach to the case of time evolving "electron" charge would be of related interest.

The value of large scale magnetic fields produced with this mechanism has been estimated. For a broad range of parameters the obtained values of the magnetic fields are much larger than the dynamo requirements.

In the present investigation the main assumption has been that the only gauge coupling free to evolve is the Abelian one. Furthermore, the parameters of the model have been chosen in such a way that the gauge coupling is not dynamical by the onset of the electroweak phase transition. It would be interesting to relax both assumptions since they may lead to potential differences with the standard scenarios.

Therefore, in many respects, the present investigation is not conclusive. At the same time it shows that acceptable models for the evolution of the gauge couplings can be obtained in a standard cosmological framework.

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